

RIEMANN-ROCH-HIRZEBRUCH INTEGRAL FORMULA FOR CHARACTERS OF REDUCTIVE LIE GROUPS

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ABSTRACT. Let $G_{\mathbb{R}}$ be a real reductive Lie group acting on a manifold M . M. Kashiwara and W. Schmid in [KaSchm] constructed representations of $G_{\mathbb{R}}$ using sheaves and quasi- $G_{\mathbb{R}}$ -equivariant \mathcal{D} -modules on M . In this article we prove an integral character formula for these representations (Theorem 1). Our main tools will be the integral localization formula recently proved in [L3] and the integral character formula proved by W. Schmid and K. Vilonen in [SchV2] (originally established by W. Rossmann in [Ro]) in the important special case when the manifold M is the flag variety of $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$ – the complexified Lie algebra of $G_{\mathbb{R}}$. In the special case when $G_{\mathbb{R}}$ is commutative and the \mathcal{D} -module is the sheaf of sections of a $G_{\mathbb{R}}$ -equivariant line bundle over M this integral character formula will reduce to the classical Riemann-Roch-Hirzebruch formula. As an illustration we give a concrete example on the enhanced flag variety.

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1. INTRODUCTION

We consider pairs of Lie groups: a real linear reductive group $G_{\mathbb{R}}$ sitting inside its complexification $G_{\mathbb{C}}$. For example:

$$\begin{array}{llll} GL(n, \mathbb{R}) & \subset & GL(n, \mathbb{C}) & SL(n, \mathbb{R}) \subset SL(n, \mathbb{C}) \\ GL^+(n, \mathbb{R}) & \subset & GL(n, \mathbb{C}) & SO(n, \mathbb{C}) \subset SL(n, \mathbb{C}) \quad Sp(n, \mathbb{R}) \subset Sp(n, \mathbb{C}). \\ U(n) & \subset & GL(n, \mathbb{C}) & SU(n) \subset SL(n, \mathbb{C}) \end{array}$$

Suppose that $G_{\mathbb{C}}$ acts algebraically on a smooth complex projective variety M . Fix a $G_{\mathbb{C}}$ -invariant open algebraic subset $U \subset M$, and take a $G_{\mathbb{C}}$ -equivariant algebraic line bundle $(\mathbf{E}, \nabla_{\mathbf{E}})$ over U with a $G_{\mathbb{C}}$ -invariant algebraic flat connection $\nabla_{\mathbf{E}}$. Let

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$S_{\mathbb{R}} \subset M$ be an open $G_{\mathbb{R}}$ -invariant subset (which may or may not be $G_{\mathbb{C}}$ -invariant) and consider the cohomology spaces

$$(1) \quad H^p(S_{\mathbb{R}}, \mathcal{O}(\mathbf{E})), \quad p \in \mathbb{Z}.$$

The classical Riemann-Roch-Hirzebruch formula computes the index of \mathbf{E} , i.e. the alternating sum

$$\sum_p (-1)^p \dim H^p(S_{\mathbb{R}}, \mathcal{O}(\mathbf{E}))$$

with $S_{\mathbb{R}} = U = M$. For general $S_{\mathbb{R}}$ and U , however, these dimensions can be infinite. To work around this problem we regard the vector spaces (1) as representations of $G_{\mathbb{R}}$, and, as a substitute for the index, we ask for the character of the virtual representation

$$\sum_p (-1)^p H^p(S_{\mathbb{R}}, \mathcal{O}(\mathbf{E})).$$

(Recall that for finite-dimensional representations the value of the character at the identity element $e \in G$ equals the dimension of the representation.)

In this article we establish an integral formula for characters of virtual representations of this kind in a more general setting of \mathcal{D} -modules. Existence of such integral character formula was conjectured by W. Schmid. My work toward this formula has led to the integral localization formula proved in [L3].

We use the following convention: whenever A is a subset of B , we denote the inclusion map $A \hookrightarrow B$ by $j_{A \hookrightarrow B}$.

2. PRELIMINARY RESULTS

Let $G_{\mathbb{C}}$ be a connected complex algebraic reductive group which is defined over \mathbb{R} . We will be primarily interested in representations of a subgroup $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ lying between the group of real points $G_{\mathbb{C}}(\mathbb{R})$ and the identity component $G_{\mathbb{C}}(\mathbb{R})^0$. We regard $G_{\mathbb{R}}$ as a real reductive Lie group. Let $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{R}}$ be the respective Lie algebras of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$. Let \mathcal{B} denote the flag variety of $\mathfrak{g}_{\mathbb{C}}$. It is a smooth complex projective variety consisting of all Borel subalgebras of $\mathfrak{g}_{\mathbb{C}}$. The group $G_{\mathbb{C}}$ acts on \mathcal{B} transitively.

If the group $G_{\mathbb{R}}$ is compact, then all irreducible representations of $G_{\mathbb{R}}$ can be enumerated by their highest weights λ lying in the weight lattice $\Lambda \subset i\mathfrak{g}_{\mathbb{R}}^*$. The Borel-Weil-Bott theorem can be regarded as an explicit construction of a holomorphic $G_{\mathbb{R}}$ -equivariant line bundle $\mathcal{L}_{\lambda} \rightarrow \mathcal{B}$ such that the resulting representation of $G_{\mathbb{R}}$ in the cohomology groups is:

$$\begin{aligned} H^p(\mathcal{B}, \mathcal{O}(\mathcal{L}_{\lambda})) &= 0 \quad \text{if } p \neq 0, \\ H^0(\mathcal{B}, \mathcal{O}(\mathcal{L}_{\lambda})) &\simeq \pi_{\lambda}, \end{aligned}$$

where $\mathcal{O}(\mathcal{L}_{\lambda})$ is the sheaf of sections of \mathcal{L}_{λ} and π_{λ} denotes the irreducible representation of $G_{\mathbb{R}}$ of highest weight λ . Then N. Berline and M. Vergne [BV], [BGV] observed that the character of π_{λ} , as a function on $\mathfrak{g}_{\mathbb{R}}$, can be expressed as an integral over \mathcal{B} of a certain naturally defined equivariantly closed form. They proved it by applying their famous integral localization formula and matching contributions from fixed points with terms of the Weyl character formula. (This is a restatement of Kirillov's character formula.)

M. Kashiwara and W. Schmid [KaSchm] generalize the Borel-Weil-Bott construction. Instead of line bundles over the flag variety \mathcal{B} they consider $G_{\mathbb{R}}$ -equivariant

sheaves¹ \mathcal{F} and, for each integer $p \in \mathbb{Z}$, they define representations of $G_{\mathbb{R}}$ in $\text{Ext}^p(\mathcal{F}, \mathcal{O}_{\mathcal{B}^{an}})$, where $\mathcal{O}_{\mathcal{B}^{an}}$ denotes the sheaf of analytic functions on \mathcal{B} . In other words, M. Kashiwara and W. Schmid prove that these vector spaces possess a natural Fréchet topology and the resulting representations of $G_{\mathbb{R}}$ are admissible of finite length. Hence these representations have characters in the sense of Harish-Chandra. Let $\tilde{\theta}$ be the character of the virtual representation of $G_{\mathbb{R}}$

$$\sum_p (-1)^p \text{Ext}^p(\mathbb{D}_{\mathcal{B}}\mathcal{F}, \mathcal{O}_{\mathcal{B}^{an}}(\lambda)),$$

where $\mathbb{D}_{\mathcal{B}}\mathcal{F}$ denotes the Verdier dual of \mathcal{F} , $\mathcal{O}_{\mathcal{B}^{an}}$ and λ is some twisting parameter lying in $\mathfrak{h}_{\mathbb{C}}^*$ – the dual space of the universal Cartan algebra of $\mathfrak{g}_{\mathbb{C}}$. Here the character θ is a distribution on $\mathfrak{g}_{\mathbb{R}}$. Then W. Schmid and K. Vilonen [SchV2] prove two character formulas for θ .

The integral character formula expresses $\tilde{\theta}$ as an integral of a certain differential form (independent of \mathcal{F}) over the characteristic cycle $Ch(\mathcal{F})$ of \mathcal{F} . Characteristic cycles were introduced by M. Kashiwara and their definition can be found in [KaScha]. On the other hand, W. Schmid and K. Vilonen give a geometric way to understand characteristic cycles in [SchV1]. A comprehensive treatment of characteristic cycles can be found in [Schü]. The cycle $Ch(\mathcal{F})$ is a conic Lagrangian Borel-Moore homology cycle lying inside the cotangent space $T^*\mathcal{B}$. If the sheaf \mathcal{F} happens to be perverse, the characteristic cycle of \mathcal{F} equals the characteristic cycle of the holonomic \mathcal{D} -module corresponding to \mathcal{F} via the Riemann-Hilbert correspondence. Originally existence of such formula over an unspecified cycle was established by W. Rossmann in [Ro]. In the special case when the group $G_{\mathbb{R}}$ is compact, the integral character formula reduces to Kirillov's character formula.

On the other hand, Harish-Chandra showed that the distribution $\tilde{\theta}$ is given by integration against a certain function $F_{\tilde{\theta}}$ on $\mathfrak{g}_{\mathbb{R}}$. This function $F_{\tilde{\theta}}$ is an $Ad(G_{\mathbb{R}})$ -invariant locally L^1 -function, its restriction to the set of regular semisimple elements $\mathfrak{g}_{\mathbb{R}}^{rs}$ of $\mathfrak{g}_{\mathbb{R}}$ can be represented by an analytic function. According to Harish-Chandra, $F_{\tilde{\theta}}$ can be expressed as follows. For a regular semisimple element $X \in \mathfrak{g}_{\mathbb{R}}^{rs}$ we denote by $\mathfrak{t}_{\mathbb{C}}(X) \subset \mathfrak{g}_{\mathbb{C}}$ the unique (complex) Cartan algebra containing X and by $\Psi(X) \subset \mathfrak{t}_{\mathbb{C}}(X)^*$ we denote the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}(X)$. Since X is regular semisimple, the number of Borel subalgebras $\{\mathfrak{b}_1, \dots, \mathfrak{b}_{|W|}\}$ containing X is exactly the order of the Weyl group W . Each Borel subalgebra $\mathfrak{b}_k \in \mathcal{B}$ containing X (and hence containing $\mathfrak{t}_{\mathbb{C}}(X)$) determines a positive root system $\Psi_{\mathfrak{b}_k}^+(X) \subset \Psi(X)$ consisting of all those roots whose root spaces are *not* contained in \mathfrak{b}_k ; that is \mathfrak{b}_k contains all negative root spaces relative to $\Psi_{\mathfrak{b}_k}^+(X)$. Then

$$(2) \quad F_{\tilde{\theta}}(X) = \sum_{k=1}^{|W|} m_{\mathfrak{b}_k}(X) \cdot \frac{e^{\langle X, \lambda_{\mathfrak{b}_k} \rangle}}{\prod_{\alpha \in \Psi_{\mathfrak{b}_k}^+(X)} \alpha(X)},$$

where $m_{\mathfrak{b}_k}(X)$'s are some integer multiplicities. W. Schmid and K. Vilonen give a formula for $m_{\mathfrak{b}_k}(X)$'s in terms of local cohomology of \mathcal{F} (formula (5.25b) in [SchV2]). Pick another positive root system $\Psi^{\leq}(X) \subset \Psi(X)$ such that

$$\text{Re } \alpha(X) \leq 0 \quad \text{for all } \alpha \in \Psi^{\leq}(X).$$

¹Strictly speaking, \mathcal{F} is not a sheaf on \mathcal{B} but rather an element of the “ $G_{\mathbb{R}}$ equivariant derived category on \mathcal{B} with twist $(-\lambda - \rho)$ ” denoted by $D_{G_{\mathbb{R}}}(\mathcal{B})_{-\lambda}$ (see [SchV2] and Remark 2).

(If there are roots in Ψ which take purely imaginary values on X , there will be different possible choices for $\Psi^{\leq}(X)$.) Let $B(X) \subset G_{\mathbb{C}}$ be the Borel subgroup whose Lie algebra contains $\mathfrak{t}_{\mathbb{C}}(X)$ and the positive root spaces corresponding to $\Psi^{\leq}(X)$. The action of $B(X)$ on \mathcal{B} has precisely $|W|$ orbits $O_1, \dots, O_{|W|}$. Each of these orbits O_k is a locally closed subset of \mathcal{B} and contains exactly one of the Borel subalgebras $\{\mathfrak{b}_1, \dots, \mathfrak{b}_{|W|}\}$ containing X . We order the Borel subalgebras containing X and the orbits of $B(X)$ so that \mathfrak{b}_k is contained in O_k , $k = 1, \dots, |W|$. Then

$$(3) \quad m_{\mathfrak{b}_k}(X) = \chi(\mathcal{H}_{O_k}^*(\mathbb{D}_{\mathcal{B}}\mathcal{F})_{\mathfrak{b}_k}) = \chi((j_{\{\mathfrak{b}_k\} \hookrightarrow O_k})^* \circ (j_{O_k \hookrightarrow \mathcal{B}})^! \mathbb{D}_{\mathcal{B}}\mathcal{F})$$

(recall that $\mathbb{D}_{\mathcal{B}}\mathcal{F}$ denotes the Verdier dual of \mathcal{F}). These multiplicities are exactly the local contributions of points $\mathfrak{b}_k \in \mathcal{B}$ to the Lefschetz fixed point formula, as generalized to sheaf cohomology by M. Goresky and R. MacPherson [GM]. W. Schmid and K. Vilonen call (2) combined with (3) the fixed point character formula because the set of all Borel subalgebras $\{\mathfrak{b}_1, \dots, \mathfrak{b}_{|W|}\}$ containing X can be expressed as the set of zeroes of the vector field generated by the infinitesimal action of X on \mathcal{B} . In the special case when the group $G_{\mathbb{R}}$ is compact, and $\lambda + \rho$ is an integral weight, all the multiplicities $m_{\mathfrak{b}_k}(X)$ are equal to each other, say,

$$m_{\mathfrak{b}_1}(X) = \dots = m_{\mathfrak{b}_{|W|}}(X) = \kappa \in \mathbb{Z},$$

and (2) reduces to κ times the Weyl character formula. The fixed point formula was conjectured by M. Kashiwara [Ka], and its proof uses the above-mentioned generalization of the Lefschetz fixed point formula to sheaf cohomology [GM].

There is a striking relationship between these two character formulas. In the compact group case N. Berline and M. Vergne [BV], [BGV] gave a simple proof of Kirillov's character formula using their integral localization formula; they matched contributions from zeroes of vector fields with terms of the Weyl character formula. However, in the non-compact group case their argument breaks down because their localization formula works for compact groups only. Originally W. Schmid and K. Vilonen [SchV2] proved these character formulas independently of each other using representation theory methods. Thus, besides an important representation-theoretical result, they formally established existence of an integral localization formula in a very special case. In the announcement [Sch] W. Schmid posed a question: "Can this equivalence of character formulas be seen directly without a detour to representation theory, just as in the compact case." In [L1], [L2] I provide such a geometric link, then in [L3] I establish a general integral localization formula for non-compact group actions. In turn, this article uses the new localization formula for non-compact group actions to give a generalization of the integral character formula to representations associated to sheaves on manifolds other than the flag manifold of $\mathfrak{g}_{\mathbb{C}}$.

We describe the integral character formula for $\tilde{\theta}$ in more detail; its ingredients will be used in our character formula (18). The character $\tilde{\theta}$ is a distribution on $\Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$ – the space of smooth compactly supported differential forms on $\mathfrak{g}_{\mathbb{R}}$ of top degree. If $\varphi \in \Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$, then we define its Fourier transform $\hat{\varphi} \in \mathcal{C}^{\infty}(\mathfrak{g}_{\mathbb{C}}^*)$ as in [L1], [L2], [L3], [Ro] and [SchV2]:

$$(4) \quad \hat{\varphi}(\xi) = \int_{\mathfrak{g}_{\mathbb{R}}} e^{\langle X, \xi \rangle} \varphi(X), \quad X \in \mathfrak{g}_{\mathbb{R}}, \xi \in \mathfrak{g}_{\mathbb{C}}^*,$$

without the customary factor of $i = \sqrt{-1}$ in the exponent.

For a smooth complex algebraic manifold M we denote by σ_M the canonical complex symplectic form on the holomorphic cotangent space T^*M . In general, when the group $G_{\mathbb{C}}$ acts on a complex manifold M , for $X \in \mathfrak{g}_{\mathbb{C}}$, we denote by X_M the vector field on M given by (notice the minus sign)

$$(5) \quad (X_M \cdot \varphi)(x) = \frac{d}{d\varepsilon} \varphi(\exp(-\varepsilon X)x) \Big|_{\varepsilon=0}, \quad \varphi \in \mathcal{C}^\infty(M).$$

Then the *moment map* on the holomorphic cotangent space $\mu_M : T^*M \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ is defined by

$$(6) \quad \mu_M(\zeta) : X \mapsto -\langle \zeta, X_M \rangle, \quad X \in \mathfrak{g}_{\mathbb{C}}, \zeta \in T^*M.$$

When the manifold M is the flag variety \mathcal{B} we get the complex algebraic symplectic form $\sigma_{\mathcal{B}}$ on $T^*\mathcal{B}$, the moment map $\mu_{\mathcal{B}} : T^*\mathcal{B} \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ and the vector field $X_{\mathcal{B}}$ for each element $X \in \mathfrak{g}_{\mathbb{C}}$.

We fix a compact real form (i.e. a maximal compact subgroup) $U_{\mathbb{R}} \subset G_{\mathbb{C}}$ with Lie algebra $\mathfrak{u}_{\mathbb{R}}$. We will use Rossmann's map $\mathcal{B} \ni x \mapsto \lambda_x \in \mathfrak{g}_{\mathbb{C}}^*$ defined in [Ro] and Section 8 of [SchV1] (here we use the notations of [SchV1] and [SchV2]). Recall that the twisting parameter λ is an element of the dual of the universal Cartan algebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$; $\mathfrak{h}_{\mathbb{C}}$ is not a subalgebra of $\mathfrak{g}_{\mathbb{C}}$, but is canonically isomorphic to any Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ equipped with a specified choice of positive root system Φ^+ . For $x \in \mathcal{B}$, we denote by $\mathfrak{b}_x \subset \mathfrak{g}_{\mathbb{C}}$ the Borel subalgebra corresponding to x . Then $\mathfrak{h}_{\mathbb{C}}$ is canonically isomorphic to the quotient $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$, so that \mathfrak{b}_x contains all the negative root spaces. Thus we have an exact sequence of vector spaces

$$(7) \quad 0 \rightarrow [\mathfrak{b}_x, \mathfrak{b}_x] \rightarrow \mathfrak{b}_x \rightarrow \mathfrak{h}_{\mathbb{C}} \rightarrow 0.$$

In general, this sequence does not have a canonical splitting. But once a choice of a compact real form $U_{\mathbb{R}} \subset G_{\mathbb{C}}$ is made, we can split this sequence as follows. Let $T_{\mathbb{R}}(x)$ be the stabilizer of x in $U_{\mathbb{R}}$, it is a maximal torus in $U_{\mathbb{R}}$, and set

$$\mathfrak{t}_{\mathbb{R}}(x) = \text{Lie}(T_{\mathbb{R}}(x)) \subset \mathfrak{u}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{C}}, \quad \mathfrak{t}_{\mathbb{C}}(x) = \mathfrak{t}_{\mathbb{R}}(x) \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}.$$

Then $\mathfrak{t}_{\mathbb{C}}(x)$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which lies in \mathfrak{b}_x and comes equipped with a system of positive roots $\Psi_{\mathfrak{b}_x}^+$ so that \mathfrak{b}_x contains all the negative root spaces. Hence we get a composition of maps

$$\mathfrak{h}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{t}_{\mathbb{C}}(x) \hookrightarrow \mathfrak{b}_x$$

which splits (7):

$$\mathfrak{b}_x \simeq \mathfrak{t}_{\mathbb{C}}(x) \oplus [\mathfrak{b}_x, \mathfrak{b}_x] \simeq \mathfrak{h}_{\mathbb{C}} \oplus [\mathfrak{b}_x, \mathfrak{b}_x].$$

We also get a splitting of $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{t}_{\mathbb{C}}(x) \oplus \left(\bigoplus \text{root spaces of } \mathfrak{t}_{\mathbb{C}}(x) \right) \simeq \mathfrak{h}_{\mathbb{C}} \oplus \left(\bigoplus \text{root spaces of } \mathfrak{t}_{\mathbb{C}}(x) \right).$$

Taking duals we get a splitting of $\mathfrak{g}_{\mathbb{C}}^*$:

$$\mathfrak{g}_{\mathbb{C}}^* \simeq \mathfrak{t}_{\mathbb{C}}(x)^* \oplus \left(\bigoplus \text{root spaces of } \mathfrak{t}_{\mathbb{C}}(x) \right)^* \simeq \mathfrak{h}_{\mathbb{C}}^* \oplus \left(\bigoplus \text{root spaces of } \mathfrak{t}_{\mathbb{C}}(x) \right)^*.$$

Hence $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ gets identified via this splitting with an element $\lambda_x \in \mathfrak{g}_{\mathbb{C}}^*$. The map $\lambda_x : \mathcal{B} \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ is smooth, real algebraic and $U_{\mathbb{R}}$ -equivariant, but in general not $G_{\mathbb{C}}$ - or $G_{\mathbb{R}}$ -equivariant, nor complex algebraic. When λ is regular, the *twisted moment map*

$$\mu_{\lambda} =_{\text{def}} \mu_{\mathcal{B}} + \lambda_x : T^*\mathcal{B} \rightarrow \mathfrak{g}_{\mathbb{C}}^*$$

is a real algebraic $U_{\mathbb{R}}$ -equivariant diffeomorphism of $T^*\mathcal{B}$ onto the coadjoint orbit of $G_{\mathbb{C}}$ of any point $\lambda_x \in \mathfrak{g}_{\mathbb{C}}^*$.

We will also use a $U_{\mathbb{R}}$ -invariant 2-form τ_{λ} on \mathcal{B} defined by the formula

$$\tau_{\lambda}(X_x, Y_x) = \lambda_x([X, Y]),$$

where $X_x, Y_x \in T_x\mathcal{B}$ are the tangent vectors at $x \in \mathcal{B}$ induced by $X, Y \in \mathfrak{u}_{\mathbb{R}}$ via differentiation of the $U_{\mathbb{R}}$ -action. It is an important property of λ_x and τ_{λ} that together they make a (non-homogeneous) differential form on \mathcal{B}

$$\langle X, \lambda_x \rangle - \tau_{\lambda}, \quad X \in \mathfrak{g}_{\mathbb{C}},$$

which depends holomorphically on X (in fact linearly) and which is *equivariantly closed* with respect to $U_{\mathbb{R}}$ (see [BGV], [GS] or [L3] for the definition of equivariantly closed forms). It follows that the forms

$$e^{\langle X, \lambda_x \rangle - \tau_{\lambda}} \quad \text{and} \quad e^{-\langle X, \lambda_x \rangle + \tau_{\lambda}}$$

are $U_{\mathbb{R}}$ -equivariantly closed forms too. For a differential form ω which is possibly non-homogeneous, we denote by $\omega_{[k]}$ its homogeneous component of degree k . Then the integral character formula (Theorem 3.8 in [SchV2]) says that the value of the character $\tilde{\theta}$ at $\varphi \in \Omega_c^{\text{top}}(\mathfrak{g}_{\mathbb{R}})$ is given by

$$\begin{aligned} (8) \quad \theta(\varphi) &= \frac{1}{(2\pi i)^d d!} \int_{Ch(\mathcal{F})} (\hat{\varphi} \circ \mu_{\lambda}) \cdot (-\sigma_{\mathcal{B}} + \tau_{\lambda})^d \\ &= \frac{1}{(-2\pi i)^d} \int_{Ch(\mathcal{F})} \left(\int_{\mathfrak{g}_{\mathbb{R}}} e^{\langle X, \mu_{\mathcal{B}}(\zeta) + \lambda_x \rangle + \sigma_{\mathcal{B}} - \tau_{\lambda}} \wedge \varphi(X) \right)_{[2d]}, \\ &\quad X \in \mathfrak{g}_{\mathbb{R}}, \zeta \in T^*\mathcal{B}, \end{aligned}$$

where $\mu_{\lambda} = \mu_{\mathcal{B}} + \lambda_x$ and $d = \dim_{\mathbb{C}} \mathcal{B}$. This integral converges because the expression $\hat{\varphi} \circ \mu_{\lambda}$ involves the Fourier transform of φ and decays rapidly along the support of $Ch(\mathcal{F})$.

3. SETUP

Recall that $G_{\mathbb{C}}$ is a connected complex algebraic reductive group which is defined over \mathbb{R} , $G_{\mathbb{C}}$ acts algebraically on a smooth complex projective variety M . And we are primarily interested in representations of a real reductive group $G_{\mathbb{R}} \subset G_{\mathbb{C}}$ lying between the group of real points $G_{\mathbb{C}}(\mathbb{R})$ and the identity component $G_{\mathbb{C}}(\mathbb{R})^0$.

We will be using the concepts of \mathcal{D} -modules and derived categories; [Bo] and [KaScha] are good introductions to these subjects. Let \mathcal{O}_M denote the structure sheaf on M and let \mathcal{D}_M denote the sheaf of linear differential operators on M with algebraic coefficients; \mathcal{D}_M acts on \mathcal{O}_M . The definition of a quasi-equivariant \mathcal{D} -module can be found in many different sources; for convenience, we copy the definition given in [KaSchm].

Let $G_{\mathbb{C}}$ be a connected complex algebraic reductive group acting algebraically on a smooth complex algebraic variety M , $\mathfrak{g}_{\mathbb{C}} = \text{Lie}(G_{\mathbb{C}})$. We write μ for the action morphism and π for the projection map $G_{\mathbb{C}} \times M \rightarrow M$:

$$\mu, \pi : G_{\mathbb{C}} \times M \rightarrow M, \quad \mu(g, x) = gx, \quad \pi(g, x) = x \quad g \in G_{\mathbb{C}}, x \in M.$$

We also consider three maps

$$\begin{aligned} q_j : G_{\mathbb{C}} \times G_{\mathbb{C}} \times M &\rightarrow G_{\mathbb{C}} \times M, \quad j = 1, 2, 3, \\ q_1(g_1, g_2, x) &= (g_1, g_2 x), \quad q_2(g_1, g_2, x) = (g_1 g_2, x), \quad q_3(g_1, g_2, x) = (g_2, x). \end{aligned}$$

Then we have the following identities:

$$\begin{aligned}\mu \circ q_1 &= \mu \circ q_2 : (g_1, g_2, x) \mapsto g_1 g_2 x, \\ \pi \circ q_2 &= \pi \circ q_3 : (g_1, g_2, x) \mapsto x, \\ \mu \circ q_3 &= \pi \circ q_1 : (g_1, g_2, x) \mapsto g_2 x.\end{aligned}$$

Definition 1. We denote by $\mathcal{O}_{G_{\mathbb{C}}} \boxtimes \mathcal{D}_M$ the subalgebra $\mathcal{O}_{G_{\mathbb{C}} \times M} \otimes_{\pi^{-1}\mathcal{O}_M} \pi^{-1}\mathcal{D}_M$ of $\mathcal{D}_{G_{\mathbb{C}} \times M}$. A quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module is a \mathcal{D}_M -module \mathfrak{M} equipped with an $\mathcal{O}_{G_{\mathbb{C}}} \boxtimes \mathcal{D}_M$ -linear isomorphism $\beta : \mu^*\mathfrak{M} \xrightarrow{\sim} \pi^*\mathfrak{M}$, such that the composition of isomorphisms

$$q_2^*\mu^*\mathfrak{M} \simeq q_1^*\mu^*\mathfrak{M} \xrightarrow{q_1^*\beta} q_1^*\pi^*\mathfrak{M} \simeq q_3^*\mu^*\mathfrak{M} \xrightarrow{q_3^*\beta} q_3^*\pi^*\mathfrak{M} \simeq q_2^*\pi^*\mathfrak{M}$$

coincides with

$$q_2^*\mu^*\mathfrak{M} \xrightarrow{q_2^*\beta} q_2^*\pi^*\mathfrak{M}.$$

If β is linear even over $\mathcal{D}_{G_{\mathbb{C}} \times M}$, this reduces to the usual definition of a $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module.

Informally speaking, this definition can be interpreted as follows. For each $g \in G_{\mathbb{C}}$, denote by $\mu_g : M \rightarrow M$ the translation by g , i.e. $\mu_g : x \mapsto gx$. Then β consists of a family of isomorphisms of \mathcal{D}_M -modules $\beta_g : \mu_g^*\mathfrak{M} \xrightarrow{\sim} \mathfrak{M}$, depending algebraically on g and multiplicative in the variable g .

Example 2. Let $(\mathbf{E}, \nabla_{\mathbf{E}})$ be a $G_{\mathbb{C}}$ -equivariant algebraic line bundle over a $G_{\mathbb{C}}$ -invariant open algebraic subset $U \subset M$ with a $G_{\mathbb{C}}$ -invariant algebraic flat connection $\nabla_{\mathbf{E}}$. Take \mathfrak{M} to be the direct image of the sheaf of sections $\mathcal{O}(\mathbf{E})$, under the inclusion map $U \hookrightarrow M$. Then \mathcal{D}_M acts on $\mathcal{O}(\mathbf{E})$ via the flat connection making $\mathcal{O}(\mathbf{E})$ a quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module.

Each element $X \in \mathfrak{g}_{\mathbb{C}}$ acts on \mathfrak{M} in two ways. One way is by inducing the vector field $-X_M$ given by (5) which in turn operates on \mathfrak{M} via the connection. The other way is by infinitesimal translation of the sections of the $G_{\mathbb{C}}$ -equivariant line bundle \mathbf{E} . When these two actions of $\mathfrak{g}_{\mathbb{C}}$ coincide, the \mathcal{D}_M -module \mathfrak{M} is $G_{\mathbb{C}}$ -equivariant.

Let \mathfrak{M} be a coherent, quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module. Recall that, for $X \in \mathfrak{g}_{\mathbb{C}}$, X_M denotes the vector field on M given by (5). Following [KaSchm] we get two different actions of $\mathfrak{g}_{\mathbb{C}}$ on \mathfrak{M} . The first action is via $\mathfrak{g}_{\mathbb{C}} \ni X \mapsto -X_M \in \Gamma(\mathcal{D}_M)$ – the global sections of \mathcal{D}_M – followed by the \mathcal{D}_M -module structure; we denote this action by $\alpha_{\mathcal{D}}$. And the second action is through differentiation of the $G_{\mathbb{C}}$ -action when we regard \mathfrak{M} as a $G_{\mathbb{C}}$ -equivariant \mathcal{O}_M -module; this action is denoted by α_t . We set $\gamma = \alpha_t - \alpha_{\mathcal{D}}$, then γ is a Lie algebra homomorphism

$$(9) \quad \gamma : \mathfrak{g}_{\mathbb{C}} \longrightarrow \text{End}_{\mathcal{D}_M}(\mathfrak{M}).$$

This way \mathfrak{M} becomes a $(\mathcal{D}_M, \mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$ -module, where $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ denotes the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. The quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module \mathfrak{M} is $G_{\mathbb{C}}$ -equivariant precisely when $\gamma \equiv 0$.

We say that the quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module \mathfrak{M} is $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite if some ideal of finite codimension $\mathcal{I} \subset \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ (the center of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$) annihilates \mathfrak{M} via the γ -action.

We denote by $Ch(\mathfrak{M})$ the characteristic cycle of \mathfrak{M} which lies in T^*M . Pick a Borel subalgebra $\mathfrak{b}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ and define a subset of $\mathfrak{g}_{\mathbb{C}}^*$

$$\mathfrak{b}_{\mathbb{C}}^{\perp} = \{\xi \in \mathfrak{g}_{\mathbb{C}}^*; \xi|_{\mathfrak{b}_{\mathbb{C}}} \equiv 0\}.$$

Then the \mathcal{D}_M -module \mathfrak{M} is called *admissible* if

$$Ch(\mathfrak{M}) \cap \mu_M^{-1}(\mathfrak{b}_\mathbb{C}^\perp) \subset T^*M$$

is a Lagrangian variety. When \mathfrak{M} is $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ -finite, the variety $Ch(\mathfrak{M}) \cap \mu_M^{-1}(\mathfrak{b}_\mathbb{C}^\perp)$ is known to be involutive ([KaMF] or [Gi]), hence the above condition is equivalent to

$$\dim_\mathbb{C}(Ch(\mathfrak{M}) \cap \mu_M^{-1}(\mathfrak{b}_\mathbb{C}^\perp)) = \dim_\mathbb{C} M.$$

Because of $G_\mathbb{C}$ -invariance of $Ch(\mathfrak{M})$, if this condition is satisfied for one particular Borel subalgebra $\mathfrak{b}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$, then it is satisfied for all Borel subalgebras of $\mathfrak{g}_\mathbb{C}$, and this definition is independent of the choice of the Borel subalgebra $\mathfrak{b}_\mathbb{C}$.

Let \mathcal{S} be a $G_\mathbb{R}$ -equivariant constructible sheaf on M . We denote by $\mathcal{O}_{M^{an}}$ the sheaf of holomorphic functions on M . Then M. Kashiwara and W. Schmid [KaSchm] equip the vector spaces

$$(10) \quad \mathrm{RHom}_{\mathcal{D}_M}^p(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{M^{an}}), \quad p \in \mathbb{Z},$$

with a natural Fréchet topology and prove that the resulting virtual representation of $G_\mathbb{R}$

$$(11) \quad \sum_p (-1)^p \mathrm{RHom}_{\mathcal{D}_M}^p(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{M^{an}})$$

is admissible of finite length whenever the \mathcal{D}_M -module \mathfrak{M} is admissible and $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ -finite. In particular, the representation (11) has a character θ in the sense of Harish-Chandra. As before, θ is a distribution on $\mathfrak{g}_\mathbb{R}$.

In this paragraph we outline M. Kashiwara and W. Schmid's construction of topology on the spaces (10). Oversimplifying and ignoring the $G_\mathbb{C}$ - and $G_\mathbb{R}$ -actions, suppose first that the sheaf \mathcal{S} is $(j_{U \hookrightarrow M})^! \mathbb{C}_U$, where $U \subset M$ is an open semi-algebraic $G_\mathbb{R}$ -invariant subset and \mathbb{C}_U is the constant sheaf on U . Furthermore, suppose that \mathfrak{M} is a *locally free* quasi- $G_\mathbb{C}$ -equivariant \mathcal{D}_M -module, i.e. $\mathfrak{M} = \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathfrak{F}$ for some coherent, locally free, $G_\mathbb{C}$ -equivariant \mathcal{O}_M -module \mathfrak{F} on M . They replace the sheaf $\mathcal{O}_{M^{an}}$ with the \mathcal{C}^∞ Dolbeault complex $\Omega_M^{(0,\cdot)}$, to which it is quasi-isomorphic, and write out isomorphisms of complexes of vector spaces without any topology:

$$\begin{aligned} (12) \quad \mathrm{RHom}_{\mathcal{D}_M}(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{M^{an}}) &\simeq \mathrm{RHom}_{\mathcal{D}_M}(\mathcal{D}_M \otimes_{\mathcal{O}_M} \mathfrak{F} \otimes (j_{U \hookrightarrow M})^! \mathbb{C}_U, \Omega_M^{(0,\cdot)}) \\ &\simeq \mathrm{RHom}_{\mathcal{O}_M}(\mathfrak{F} \otimes (j_{U \hookrightarrow M})^! \mathbb{C}_U, \Omega_M^{(0,\cdot)}) \simeq \mathrm{R}\Gamma(U; (\mathfrak{F}^*)^{an} \otimes_{\mathcal{O}_{M^{an}}} \Omega_M^{(0,\cdot)}) \\ &\simeq \Gamma(U; (\mathfrak{F}^*)^{an} \otimes_{\mathcal{O}_{M^{an}}} \Omega_M^{(0,\cdot)}), \end{aligned}$$

where $(\mathfrak{F}^*)^{an}$ denotes the sheaf of analytic sections of the dual of \mathfrak{F} . The complex on the right has a natural Fréchet topology – the \mathcal{C}^∞ topology for differential forms – and continuous $G_\mathbb{R}$ action. General \mathfrak{M} and \mathcal{S} , have resolutions by locally free quasi- $G_\mathbb{C}$ -equivariant \mathcal{D}_M -modules (Lemma 4.7 in [KaSchm]) and by sheaves of the type $(j_{U \hookrightarrow M})^! \mathbb{C}_U$ respectively. This, combined with the acyclicity of the right hand side of (12), makes it possible to equip the vector spaces (10) with a Fréchet topology. Then M. Kashiwara and W. Schmid work hard to show that this topology does not depend on the choices involved. They do it by proving that the topology on the spaces (10) is that of the maximal globalization of their underlying Harish-Chandra modules.

Recall that \mathcal{B} denotes the flag variety of $\mathfrak{g}_{\mathbb{C}}$. We will establish a formula for this character as an integral over a cycle in $T^*(\mathcal{B} \times M)$ under an additional assumption that \mathfrak{M} has an infinitesimal character, i.e. $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ acts on \mathfrak{M} by a character. Even if this condition is not satisfied, since \mathfrak{M} is assumed to be $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite, there is a finite filtration of \mathfrak{M} by quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -submodules such that the successive quotients have infinitesimal characters, and we can apply this integral character formula to each of these quotients separately.

In the special case when $G_{\mathbb{C}} = \mathbb{C}^{\times}$, the \mathcal{D}_M -module \mathfrak{M} is the sheaf of sections of a $G_{\mathbb{C}}$ -equivariant line bundle $(\mathbf{E}, \nabla_{\mathbf{E}})$ over M as in Example 2 with $U = M$, and \mathcal{S} is the constant sheaf \mathbb{C}_M , the flag variety \mathcal{B} consists of just one point and the integral character formula will reduce to the classical Riemann-Roch-Hirzebruch formula.

Remark 1. *We do not need M to be projective to establish that (11) is admissible of finite length. In fact it is sufficient to assume that M is a smooth quasi-projective variety. The compactness of M will be needed for the integral character formula.*

On the other hand, the result of Sumihiro [Su] restated as Proposition 4.6 in [KaSchm] together with Theorem 5.12 there show that we can always embed M into a smooth projective variety and there is no loss of generality in assuming that M is projective. We will illustrate this in Section 6.

Our derivation of the integral character formula for θ follows the following scheme:

- Replace the pair $(\mathfrak{M}, \mathcal{S})$ which lives on M with a new pair $(\widetilde{\mathfrak{M}}, \widetilde{\mathcal{S}})$ which lives on the flag variety \mathcal{B} such that the virtual representation (11) stays unchanged (14);
- Write out the fixed point character formula for θ in terms of $(\widetilde{\mathfrak{M}}, \widetilde{\mathcal{S}})$ and zeroes on \mathcal{B} ;
- We want to prove that the integral (8) represents θ ; first we apply the integral localization formula [L3] to (8);
- Combine similar terms in the result of the previous step and match them with the terms in the fixed point character formula for θ obtained in the earlier step.

4. STATEMENT OF THE MAIN RESULT

In the previous section we assumed that \mathfrak{M} was a quasi- $G_{\mathbb{C}}$ -equivariant coherent \mathcal{D}_M -module on M which was admissible and had an infinitesimal character. We will follow [KaSchm] and index characters of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ by linear functionals λ on the universal Cartan without the customary shift by ρ (half sum of the positive roots); in other words, $\chi_{\lambda} : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathbb{C}$ denotes the character by which $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ acts on the Verma module with highest weight λ . Then $\chi_{\lambda} = \chi_{\mu}$ if and only if $\lambda + \rho$ is conjugate to $\mu + \rho$ under the action of the Weyl group W .

Remark 2. *In this article we use results from [KaSchm] and [SchV2]. Unfortunately these two sources use different conventions for labeling characters and twists. We follow the conventions of [KaSchm] explained above. On the other hand [SchV2] shift these notations by ρ and they define, for instance, the “ $G_{\mathbb{R}}$ -equivariant derived category on \mathcal{B} with twist $(\lambda - \rho)$ ” denoted by $D_{G_{\mathbb{R}}}(\mathcal{B})_{\lambda}$ which becomes the bounded $G_{\mathbb{R}}$ -equivariant derived category in the usual “untwisted” sense precisely when $\lambda = \rho$. Similarly, $\mathcal{O}_{\mathcal{B}}(\lambda)$ in [SchV2] denotes the twisted sheaf of holomorphic*

functions with twist $(\lambda - \rho)$ and which becomes the sheaf of ordinary holomorphic functions precisely when $\lambda = \rho$.

We assume that \mathfrak{M} is an object in $\text{Mod}_{\text{quasi-}G_{\mathbb{C}}\text{-eq}}^{\text{coh}, \lambda-\rho}(\mathcal{D}_M)$ – the category of quasi- $G_{\mathbb{C}}$ -equivariant coherent \mathcal{D}_M -modules on M with infinitesimal character $\chi_{\lambda-\rho}$.

The Kashiwara-Schmid construction of Fréchet topology on the spaces (10) was carried out on the level of derived categories. This means that instead of a sheaf \mathcal{S} we have an element $\mathcal{S} \in D_{G_{\mathbb{R}}, \mathbb{R}\text{-}c}^b(\mathbb{C}_M)$ – the bounded derived category of $G_{\mathbb{R}}$ -equivariant \mathbb{R} -constructible sheaves, and we have a pairing

$$\begin{aligned} \text{Mod}_{\text{quasi-}G_{\mathbb{C}}\text{-eq}}^{\text{coh}, \lambda-\rho}(\mathcal{D}_M) \times D_{G_{\mathbb{R}}, \mathbb{R}\text{-}c}^b(\mathbb{C}_M) &\longrightarrow D^b(\mathcal{F}_{G_{\mathbb{R}}}), \\ (\mathfrak{M}, \mathcal{S}) &\longmapsto \text{RHom}_{\mathcal{D}_M}^{\text{top}}(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{M^{\text{an}}}), \end{aligned}$$

where $D^b(\mathcal{F}_{G_{\mathbb{R}}})$ denotes the derived category of $G_{\mathbb{R}}$ -representations defined in §3 of [KaSchm]. The category $D^b(\mathcal{F}_{G_{\mathbb{R}}})$ is built on complexes of topological vector spaces with continuous $G_{\mathbb{R}}$ -actions (C, d_C) , the differential maps $d_C^n : C^n \rightarrow C^{n+1}$ are required to be continuous and $G_{\mathbb{R}}$ -equivariant. A complex (C, d_C) is *exact* if, for all n ,

$$d_C^n : C^n \rightarrow \ker d_C^{n+1}$$

is onto and is an open map relative to the subspace topology on $\ker d_C^{n+1}$. A complex (C, d_C) becomes zero in $D^b(\mathcal{F}_{G_{\mathbb{R}}})$ precisely when it is exact.

Because we will apply results of A. Beilinson and J. Bernstein on equivalences of categories [BB], we will also assume that λ is *integrally dominant*, i.e.

$$\langle \check{\alpha}, \lambda \rangle \notin \mathbb{Z}_{<0}, \quad \text{for every positive coroot } \check{\alpha}.$$

This can always be achieved by replacing λ with an appropriate W -translate.

We denote by $\text{Mod}_{G_{\mathbb{C}}\text{-eq}}(\mathcal{D}_{\mathcal{B}, \lambda-\rho})$ the category of modules over the sheaf of twisted differential operators $\mathcal{D}_{\mathcal{B}, \lambda-\rho}$ on the flag variety \mathcal{B} . The sheaf $\mathcal{D}_{\mathcal{B}, \lambda-\rho}$ is defined in [BB], but we follow the twisting conventions of [KaSchm] as explained in Remark 2, so that $\mathcal{D}_{\mathcal{B}, 0} = \mathcal{D}_{\mathcal{B}}$ – the sheaf of differential operators on \mathcal{B} without any twist. We form a product space $\mathcal{B} \times M$ with diagonal $G_{\mathbb{C}}$ -action and consider the sheaf of twisted differential operators $\mathcal{D}_{\mathcal{B} \times M, \lambda-\rho} =_{\text{def}} \mathcal{D}_{\mathcal{B}, \lambda-\rho} \boxtimes \mathcal{D}_M$, the twisting is confined to the factor \mathcal{B} . We also denote by $\text{Mod}_{G_{\mathbb{C}}\text{-eq}}(\mathcal{D}_{\mathcal{B} \times M, \lambda-\rho})$ the category of $G_{\mathbb{C}}$ -equivariant $\mathcal{D}_{\mathcal{B} \times M, \lambda-\rho}$ -modules on $\mathcal{B} \times M$. Let p and q be the projection maps

$$\begin{array}{ccc} \mathcal{B} & \xleftarrow{p} & \mathcal{B} \times M \\ & & \downarrow q \\ & & M, \end{array}$$

and let \tilde{p} and \tilde{q} be the induced projections on the cotangent bundles:

$$\begin{array}{ccc} T^*\mathcal{B} & \xleftarrow{\tilde{p}} & T^*(\mathcal{B} \times M) \\ & & \downarrow \tilde{q} \\ & & T^*M. \end{array}$$

M. Kashiwara and W. Schmid [KaSchm] use results of A. Beilinson and J. Bernstein on equivalence of categories [BB] to prove that the pair $(\mathfrak{M}, \mathcal{S})$ on M can be replaced with a $\mathcal{D}_{\mathcal{B}, \rho-\lambda}$ -module and a complex of sheaves on \mathcal{B} . We will make this

statement precise. First of all, they show that there exists a coherent holonomic module $\mathfrak{L} \in \text{Mod}_{G_{\mathbb{C}}-eq}(\mathcal{D}_{\mathcal{B} \times M, \lambda-\rho})$ such that

$$q_* \mathfrak{L} = \mathfrak{M} \quad \text{and} \quad R^k q_* \mathfrak{L} = 0 \text{ if } k \neq 0,$$

namely one can take

$$\mathfrak{L} = \mathcal{D}_{\mathcal{B}, \lambda-\rho} \boxtimes \mathfrak{M} / \gamma_{\mathcal{B} \times M}(\mathfrak{g}_{\mathbb{C}})(\mathcal{D}_{\mathcal{B}, \lambda-\rho} \boxtimes \mathfrak{M}),$$

where $\gamma_{\mathcal{B} \times M}(\mathfrak{g}_{\mathbb{C}})(\mathcal{D}_{\mathcal{B}, \lambda-\rho} \boxtimes \mathfrak{M})$ denotes the image in $\mathcal{D}_{\mathcal{B}, \lambda-\rho} \boxtimes \mathfrak{M}$ of the module $\mathfrak{g}_{\mathbb{C}} \otimes (\mathcal{D}_{\mathcal{B}, \lambda-\rho} \boxtimes \mathfrak{M})$ under the map $\gamma_{\mathcal{B} \times M}$ given by the equation (9) with the ambient manifold $\mathcal{B} \times M$.

Then we apply the twisted deRham functor to obtain a complex of sheaves

$$\mathcal{L} = DR_{\mathcal{B} \times M}(\mathfrak{L}) =_{def} R\mathcal{H}om_{\mathcal{D}_{\mathcal{B} \times M, \lambda-\rho}}(\mathcal{O}_{\mathcal{B}}(\lambda - \rho) \boxtimes \mathcal{O}_M, \mathfrak{L}),$$

which is an element of the bounded $G_{\mathbb{C}}$ -equivariant, \mathbb{C} -constructible derived category with twist $(\rho - \lambda)$ along the \mathcal{B} -factor denoted $D_{G_{\mathbb{C}}, \rho-\lambda, \mathbb{C}-c}^b(\mathbb{C}_{\mathcal{B} \times M})$. Here $\mathcal{O}_{\mathcal{B}}(\lambda - \rho)$ is a twisted sheaf of holomorphic functions on \mathcal{B} , with twist $(\lambda - \rho)$ (so that $\mathcal{O}_{\mathcal{B}}(0)$ is just the sheaf of functions on \mathcal{B} with no twist at all).

Let $d = \dim_{\mathbb{C}} \mathcal{B}$, and let $(\Omega_{\mathcal{B}}^d)^{-1} \simeq \mathcal{O}_{\mathcal{B}}(2\rho)$ denote the reciprocal of the canonical sheaf. Combining the equations (6.7), (7.14) and (7.15) from [KaSchm] we obtain

$$\begin{aligned} (13) \quad & R\mathcal{H}om_{\mathcal{D}_M}^{top}(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{M^{an}}) \\ & \simeq R\mathcal{H}om_{\mathcal{D}_{\mathcal{B}, \rho-\lambda}}^{top}(\mathcal{D}_{\mathcal{B}, \rho-\lambda} \otimes \mathcal{O}_{\mathcal{B}}(\Omega_{\mathcal{B}}^d)^{-1} \otimes Rp_*(\mathcal{L} \otimes q^{-1}\mathcal{S}), \mathcal{O}_{\mathcal{B}^{an}}(\rho - \lambda))[d - 2 \dim_{\mathbb{C}} M] \\ & \simeq R\mathcal{H}om_{\mathcal{D}_{\mathcal{B}, -\lambda-\rho}}^{top}(\mathcal{D}_{\mathcal{B}, -\lambda-\rho} \otimes Rp_*(\mathcal{L} \otimes q^{-1}\mathcal{S}), \mathcal{O}_{\mathcal{B}^{an}}(-\lambda - \rho))[d - 2 \dim_{\mathbb{C}} M] \\ & \simeq R\mathcal{H}om(Rp_*(\mathcal{L} \otimes q^{-1}\mathcal{S}), \mathcal{O}_{\mathcal{B}^{an}}(-\lambda - \rho))[d - 2 \dim_{\mathbb{C}} M] \end{aligned}$$

as elements of the derived category of $G_{\mathbb{R}}$ -representations $D^b(\mathcal{F}_{G_{\mathbb{R}}})$. Here we view $Rp_*(\mathcal{L} \otimes q^{-1}\mathcal{S})$ as an object in $D_{G_{\mathbb{R}}, -\lambda-\rho, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B}})$, which makes sense because 2ρ is an integral weight, and this implies the existence of a canonical isomorphism (6.8) in [KaSchm]

$$D_{G_{\mathbb{R}}, \rho-\lambda, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B}}) \simeq D_{G_{\mathbb{R}}, -\lambda-\rho, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B}}).$$

That is, the pair $(\mathfrak{M}, \mathcal{S})$ on the variety M is replaced by a pair

$$(14) \quad (\mathcal{D}_{\mathcal{B}, -\lambda-\rho}, Rp_*(\mathcal{L} \otimes q^{-1}\mathcal{S}))$$

on the flag variety \mathcal{B} , with an additional twist by $(-\lambda - \rho)$.

Our starting point is the integral character formula (8) proved by W. Schmid and K. Vilonen in [SchV2]. We apply it to the right hand side of (13). We fix a compact real form $U_{\mathbb{R}} \subset G_{\mathbb{C}}$. Recall that $\mu_{\mathcal{B}} : T^*\mathcal{B} \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ is the moment map defined by (6), $\sigma_{\mathcal{B}}$ is the canonical complex algebraic holomorphic symplectic form on $T^*\mathcal{B}$, $\lambda_x : \mathcal{B} \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ is the $U_{\mathbb{R}}$ -equivariant Rossmann's map, and τ_{λ} is a certain $U_{\mathbb{R}}$ -invariant 2-form on \mathcal{B} . The character of the virtual representation (11) is a distribution on $\Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$ – the space of smooth compactly supported differential forms on $\mathfrak{g}_{\mathbb{R}}$ of top degree. For an element $\varphi \in \Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$, its Fourier transform $\hat{\varphi} \in \mathcal{C}^{\infty}(\mathfrak{g}_{\mathbb{C}}^*)$ is defined by (4) without the customary factor of $i = \sqrt{-1}$ in the exponent. Then the integral character formula says that the character θ of the virtual representation (13) of $G_{\mathbb{R}}$,

as a distribution on $\Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$, is

$$(15) \quad \theta(\varphi) = \frac{1}{(2\pi i)^d d!} \int_{Ch(Rp_*(\mathcal{L} \otimes q^{-1}S))^a} (\hat{\varphi} \circ \mu_{-\lambda}) \cdot (\sigma_{\mathcal{B}} + \tau_{\lambda})^d \\ = \frac{1}{(2\pi i)^d} \int_{Ch(Rp_*(\mathcal{L} \otimes q^{-1}S))^a} \left(\int_{\mathfrak{g}_{\mathbb{R}}} e^{\langle X, \mu_{\mathcal{B}}(\zeta) - \lambda_x \rangle + \sigma_{\mathcal{B}} + \tau_{\lambda}} \wedge \varphi(X) \right)_{[2d]}, \\ X \in \mathfrak{g}_{\mathbb{R}}, \zeta \in T^*\mathcal{B}.$$

Here $\mu_{-\lambda} = \mu_{\mathcal{B}} - \lambda_x$, $d = \dim_{\mathbb{C}} \mathcal{B}$, $a : T^*\mathcal{B} \rightarrow T^*\mathcal{B}$ is the antipodal map $\zeta \mapsto -\zeta$, and $Ch(Rp_*(\mathcal{L} \otimes q^{-1}S))^a$ denotes the image under this antipodal map of the characteristic cycle of $Rp_*(\mathcal{L} \otimes q^{-1}S)$ (which is a cycle in $T^*\mathcal{B}$).

Remark 3. *If Z is a complex manifold and $Z^{\mathbb{R}}$ is the underlying real analytic manifold, then the holomorphic symplectic form σ_Z is defined on the holomorphic cotangent bundle T^*Z , while the characteristic cycles of constructible sheaves on Z lie in the real cotangent bundle $T^*(Z^{\mathbb{R}})$. Hence we need to identify T^*Z with $T^*(Z^{\mathbb{R}})$. There are at least two different but equally natural ways of doing this, we use the convention (11.1.2) of [KaScha], Chapter XI; the same convention is used in [L1], [L2], [L3] and [SchV2]. Under this convention, if $\sigma_{Z, \mathbb{R}}$ is the canonical real symplectic form on $T^*Z^{\mathbb{R}}$ and σ_Z is the canonical complex holomorphic symplectic form on T^*Z , then $\sigma_{Z, \mathbb{R}}$ gets identified with $2 \operatorname{Re} \sigma_Z$.*

Another important ingredient is a generalization of the Hopf index theorem stated as Corollary 9.5.2 in [KaScha]. Let \mathcal{T} be a constructible sheaf on M or an element of the derived category $D_{\mathbb{R}-c}^b(\mathbb{C}_M)$, and let $\chi(M, \mathcal{T})$ denote the Euler characteristic of M with respect to \mathcal{T} . Then

$$(16) \quad \chi(M, \mathcal{T}) = \#([M] \cap Ch(\mathcal{T})),$$

where $[M]$ denotes the fundamental cycle of M . Alternatively one can apply the equation (5.30) from [Schü]. Let $\operatorname{Thom}_{T^*M}$ denote the Thom form of the cotangent bundle $T^*M \rightarrow M$. That is $\operatorname{Thom}_{T^*M}$ is a closed differential form on T^*M of degree $2 \dim_{\mathbb{C}} M$ which decays rapidly along the fiber (or even compactly supported along the fiber) and such that

$$\int_{T_x^*M} \operatorname{Thom}_{T^*M} = (2\pi)^{\dim_{\mathbb{C}} M}, \quad \forall x \in M.$$

We regard M as a submanifold of T^*M via the zero section inclusion. Then the restriction of the Thom form to M is the Euler form of M . Since the form $(2\pi)^{-\dim_{\mathbb{C}} M} \operatorname{Thom}_{T^*M}$ is Poincaré dual to the homology class of $[M]$ in T^*M , the Hopf index theorem (16) can be rewritten as

$$(17) \quad \chi(M, \mathcal{T}) = \#([M] \cap Ch(\mathcal{T})) = (2\pi)^{-\dim_{\mathbb{C}} M} \int_{Ch(\mathcal{T})} \operatorname{Thom}_{T^*M},$$

which is a generalization of the Gauss-Bonnet theorem.

Recall that $U_{\mathbb{R}}$ is a compact real form of $G_{\mathbb{C}}$. The form $\operatorname{Thom}_{T^*M}$ may be chosen to be $U_{\mathbb{R}}$ -invariant. If, in addition, the cotangent bundle T^*M has a spin structure, then V. Mathai and D. Quillen showed in [MQ] (see also Section 7.7 in [BGV]) that $\operatorname{Thom}_{T^*M}$ can be realized as the top degree part of a $U_{\mathbb{R}}$ -equivariantly closed form on T^*M in a canonical way.

The last essential ingredient is the deformation argument for integrals of equivariantly closed forms from [L3]. This argument requires that any maximal complex torus $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ acts on M with isolated fixed points. (Since M is compact, there will be only finitely many of those.) If this condition is satisfied for one particular torus, then it is satisfied for all tori because all maximal complex tori are conjugate by elements of $G_{\mathbb{C}}$. (This condition is satisfied in all application we have in mind.)

We will combine the integral character formula (15) and the Gauss-Bonnet formula (17) to get a formula for the character of the virtual representation (13) as an integral over a cycle in $T^*(\mathcal{B} \times M)$. Set

$$\tilde{\mathcal{S}} = \mathcal{L} \otimes q^{-1}\mathcal{S} \quad \text{and} \quad \Lambda = Ch(\tilde{\mathcal{S}})^a = Ch(\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}})) \subset T^*(\mathcal{B} \times M).$$

Here $a : T^*(\mathcal{B} \times M) \rightarrow T^*(\mathcal{B} \times M)$ is the antipodal map $\zeta \mapsto -\zeta$ and $Ch(\tilde{\mathcal{S}})^a$ denotes the image under this antipodal map of the characteristic cycle of $\tilde{\mathcal{S}}$; the operator $\mathbb{D}_{\mathcal{B} \times M}$ is the Verdier duality operator (see (22) below for its properties).

Theorem 1. *Suppose that any maximal complex torus $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ acts on M with isolated fixed points. Then the value of the character θ of the virtual representation (11) on $\varphi \in \Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$ is*

$$(18) \quad \begin{aligned} \theta(\varphi) &= \frac{i^n}{(2\pi i)^{d+n} d!} \int_{\Lambda} \tilde{p}^* ((\hat{\varphi} \circ \mu_{-\lambda}) \cdot (\sigma_{\mathcal{B}} + \tau_{\lambda})^d) \wedge \tilde{q}^* \text{Thom}_{T^*M} \\ &= \frac{i^n}{(2\pi i)^{d+n}} \int_{\Lambda} \left(\int_{\mathfrak{g}_{\mathbb{R}}} \tilde{p}^* e^{\langle X, \mu_{\mathcal{B}}(\zeta) - \lambda_x \rangle + \sigma_{\mathcal{B}} + \tau_{\lambda}} \wedge \tilde{q}^* \text{Thom}_{T^*M} \wedge \varphi(X) \right)_{[2(d+n)]}, \end{aligned}$$

where $\mu_{-\lambda} = \mu_{\mathcal{B}} - \lambda_x$, $d = \dim_{\mathbb{C}} \mathcal{B}$ and $n = \dim_{\mathbb{C}} M$.

Remark 4. *Suppose that the group $G_{\mathbb{C}} = \mathbb{C}^{\times}$, the \mathcal{D}_M -module \mathfrak{M} is the sheaf of sections of a $G_{\mathbb{C}}$ -equivariant line bundle $(\mathbf{E}, \nabla_{\mathbf{E}})$ over M as in Example 2, and \mathcal{S} is the constant sheaf \mathbb{C}_M . Then the flag variety \mathcal{B} is just one point, the cycle $\Lambda = [M]$, and because of the Riemann-Roch relationship (8.4) in [MQ] (or Theorem 7.44 in [BGV]) the above integral character formula will reduce to the classical Riemann-Roch-Hirzebruch formula. There is no curvature of the line bundle $(\mathbf{E}, \nabla_{\mathbf{E}})$ present in the character formula (18) because we assume that the connection $\nabla_{\mathbf{E}}$ is flat so that $\mathcal{O}(\mathbf{E})$ is a quasi- $G_{\mathbb{C}}$ -equivariant \mathcal{D}_M -module.*

5. PROOF OF THEOREM 1

We start our proof by applying the fixed point character formula (2) combined with (3) due to W. Schmid and K. Vilonen [SchV2] to the right hand side of (13). Thus the character θ of the virtual representation (13) is given by integration against a function F_{θ} on $\mathfrak{g}_{\mathbb{R}}$:

$$(19) \quad \theta(\varphi) = \int_{\mathfrak{g}_{\mathbb{R}}} F_{\theta} \varphi, \quad \varphi \in \Omega_c^{top}(\mathfrak{g}_{\mathbb{R}}).$$

This function F_{θ} is an $Ad(G_{\mathbb{R}})$ -invariant, locally L^1 function on $\mathfrak{g}_{\mathbb{R}}$ whose restriction to the set of regular semisimple elements $\mathfrak{g}_{\mathbb{R}}^{rs}$ can be represented by a real analytic function. The value of this analytic function at $X \in \mathfrak{g}_{\mathbb{R}}^{rs}$ is determined by zeroes of the vector field $X_{\mathcal{B}}$ on the flag variety \mathcal{B} as follows.

Recall that $\mathfrak{t}_{\mathbb{C}}(X) \subset \mathfrak{g}_{\mathbb{C}}$ is the unique Cartan subalgebra containing $X \in \mathfrak{g}_{\mathbb{R}}^{rs}$. Let $\Psi(X) \subset \mathfrak{t}_{\mathbb{C}}^*(X)$ be the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}(X)$ and pick a positive

root system $\Psi^{\leq}(X) \subset \Psi$ such that

$$\operatorname{Re} \alpha(X) \leq 0 \quad \text{for all } \alpha \in \Psi^{\leq}(X).$$

Let $B(X) \subset G_{\mathbb{C}}$ be the Borel subgroup whose Lie algebra contains $\mathfrak{t}_{\mathbb{C}}(X)$ and the positive root spaces corresponding to $\Psi^{\leq}(X)$. The action of $B(X)$ on \mathcal{B} has exactly the same number of orbits $O_1, \dots, O_{|W|}$ as the order of the Weyl group W of $G_{\mathbb{C}}$. Each of these orbits O_k contains exactly one zero of the vector field $X_{\mathcal{B}}$, and we order the zeroes $\{\mathfrak{b}_1, \dots, \mathfrak{b}_{|W|}\}$ of $X_{\mathcal{B}}$ so that \mathfrak{b}_k is contained in O_k , $k = 1, \dots, |W|$. The set of zeroes $\{\mathfrak{b}_1, \dots, \mathfrak{b}_{|W|}\}$ is precisely the set of Borel subalgebras containing $\mathfrak{t}_{\mathbb{C}}(X)$. Let $\Psi_{\mathfrak{b}_k}^+(X) \subset \Psi(X)$ be the positive root system such that \mathfrak{b}_k contains all the negative root spaces, $k = 1, \dots, |W|$. Then the fixed point character formula [SchV2] says that the function F_{θ} which appeared in the equation (19) is

$$(20) \quad F_{\theta}(X) = (-1)^d \sum_{k=1}^{|W|} m_{\mathfrak{b}_k}(X) \frac{e^{-\langle X, \lambda_{\mathfrak{b}_k} \rangle}}{\prod_{\alpha \in \Psi_{\mathfrak{b}_k}^+(X)} \alpha(X)},$$

where $m_{\mathfrak{b}_k}(X)$'s are integer multiplicities given by the formula

$$(21) \quad m_{\mathfrak{b}_k}(X) = \chi(\mathcal{H}_{O_k}^*(Rp_*(\tilde{\mathcal{S}}))_{\mathfrak{b}_k}) = \chi((j_{\{\mathfrak{b}_k\} \hookrightarrow O_k})^* \circ (j_{O_k \hookrightarrow \mathcal{B}})^!(Rp_*(\tilde{\mathcal{S}}))).$$

Let $\mathbb{D}_{\mathcal{B}}$ and $\mathbb{D}_{\mathcal{B} \times M}$ denote the Verdier duality operators on \mathcal{B} and $\mathcal{B} \times M$ respectively:

$$\mathbb{D}_{\mathcal{B}} : \quad D_{G_{\mathbb{R}}, -\lambda-\rho, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B}}) \xrightarrow{\sim} D_{G_{\mathbb{R}}, \lambda+\rho, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B}});$$

$$(22) \quad \mathbb{D}_{\mathcal{B} \times M} : \quad D_{G_{\mathbb{R}}, -\lambda-\rho, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B} \times M}) \xrightarrow{\sim} D_{G_{\mathbb{R}}, \lambda+\rho, \mathbb{R}-c}^b(\mathbb{C}_{\mathcal{B} \times M}).$$

The effect of the Verdier duality operator \mathbb{D}_Z on the characteristic cycle of an \mathbb{R} -constructible sheaf \mathcal{T} (or an element of $D_{\mathbb{R}-c}^b(\mathbb{C}_Z)$) on any smooth quasi-projective variety Z is described by

$$Ch(\mathbb{D}_Z(\mathcal{T})) = Ch(\mathcal{T})^a,$$

where $a : T^*Z \rightarrow T^*Z$ is the antipodal map $\zeta \mapsto -\zeta$ and $Ch(\mathcal{T})^a$ denotes the image under this antipodal map of the characteristic cycle of \mathcal{T} .

For a regular semisimple element $X \in \mathfrak{g}_{\mathbb{C}}^{rs}$ we denote by $T_{\mathbb{C}}(X) = \exp(\mathfrak{t}_{\mathbb{C}}(X)) \subset G_{\mathbb{C}}$ the maximal complex torus corresponding to the unique Cartan subalgebra containing X . If $x \in M$ is a point fixed by $T_{\mathbb{C}}(X)$; then $T_{\mathbb{C}}(X)$ acts linearly on the tangent space $T_x M$. We denote by $\Delta(X) \subset \mathfrak{t}_{\mathbb{C}}(X)^*$ the set of weights of $\mathfrak{t}_{\mathbb{C}}(X)$ which are either roots of $\mathfrak{g}_{\mathbb{C}}$ or occur in the tangent space $T_x M$ of some point $x \in M$ fixed by $T_{\mathbb{C}}(X)$.

Let $\mathfrak{g}'_{\mathbb{R}}$ denote the set of *strongly regular semisimple* elements $X \in \mathfrak{g}_{\mathbb{R}}^{rs}$ which satisfy the following additional properties. If $\mathfrak{t}_{\mathbb{R}}(X) \subset \mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{t}_{\mathbb{C}}(X) \subset \mathfrak{g}_{\mathbb{C}}$ are the unique Cartan subalgebras in $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{C}}$ respectively containing X , then:

- (1) The set of zeroes of the vector field X_M is exactly the set of points in M fixed by the complex torus $T_{\mathbb{C}}(X) = \exp(\mathfrak{t}_{\mathbb{C}}(X)) \subset G_{\mathbb{C}}$;
- (2) $\beta(X) \neq 0$ for all $\beta \in \Delta(X) \subset \mathfrak{t}_{\mathbb{C}}(X)^*$;
- (3) For each $\beta \in \Delta(X)$, we have either

$$(23) \quad \operatorname{Re}(\beta)|_{\mathfrak{t}_{\mathbb{R}}(X)} \equiv 0 \quad \text{or} \quad \operatorname{Re}(\beta(X)) \neq 0.$$

Clearly, $\mathfrak{g}'_{\mathbb{R}}$ is an open subset of $\mathfrak{g}_{\mathbb{R}}$; since M is compact and $\Delta(X)$ is finite, the complement of $\mathfrak{g}'_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ has measure zero.

From now on we will assume that the element $X \in \mathfrak{g}_{\mathbb{R}}$ is not only regular semisimple, but also lies in $\mathfrak{g}'_{\mathbb{R}}$. Then applying the integral localization formula of [L3] to the integral character formula (15), with the global formula for the multiplicities, we can rewrite the formula (21) as

$$(24) \quad m_{\mathfrak{b}_k}(X) = \chi(\mathcal{B}, (\mathbb{D}_{\mathcal{B}}(Rp_*(\tilde{\mathcal{S}})))_{O_k}) = \chi(\mathcal{B}, (j_{O_k \hookrightarrow \mathcal{B}})_! \circ (j_{O_k \hookrightarrow \mathcal{B}})^*(\mathbb{D}_{\mathcal{B}}(Rp_*(\tilde{\mathcal{S}}))))).$$

Applying Proposition 2.5.11 from [KaScha] to the Cartesian square

$$\begin{array}{ccc} O_k \times M & \hookrightarrow & \mathcal{B} \times M \\ \downarrow & & \downarrow \\ O_k & \hookrightarrow & \mathcal{B} \end{array}$$

and using that the projection map p is proper we obtain:

$$\begin{aligned} (25) \quad m_{\mathfrak{b}_k}(X) &= \chi(\mathcal{B}, (j_{O_k \hookrightarrow \mathcal{B}})_! \circ (j_{O_k \hookrightarrow \mathcal{B}})^*(Rp_!(\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}})))) \\ &= \chi(\mathcal{B}, (j_{O_k \hookrightarrow \mathcal{B}})_! \circ Rp_! \circ (j_{O_k \times M \hookrightarrow \mathcal{B} \times M})^*(\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))) \\ &= \chi(\mathcal{B}, Rp_! \circ (j_{O_k \times M \hookrightarrow \mathcal{B} \times M})_! \circ (j_{O_k \times M \hookrightarrow \mathcal{B} \times M})^*(\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))) \\ &= \chi(\mathcal{B} \times M, (j_{O_k \times M \hookrightarrow \mathcal{B} \times M})_! \circ (j_{O_k \times M \hookrightarrow \mathcal{B} \times M})^*(\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))) \\ &= \chi(\mathcal{B} \times M, (\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))_{O_k \times M}). \end{aligned}$$

Next we will compare this result with the result of application of the integral localization formula of [L3] to the integral (18). Let

$$M = \coprod_{\{x \in M; X_M(x)=0\}} \tilde{O}_x$$

be the Bialynicki-Birula decomposition [Bi] of M into attracting sets (relative to X), as described in [L3]. To obtain this decomposition we pick any $X' \in \mathfrak{t}_{\mathbb{R}}(X) \cap \mathfrak{g}'_{\mathbb{R}}$ in the same connected component of $\mathfrak{t}_{\mathbb{R}}(X) \cap \mathfrak{g}'_{\mathbb{R}}$ and such that

$$\operatorname{Re} \beta(X) > 0 \iff \operatorname{Re} \beta(X') > 0 \quad \text{and} \quad \operatorname{Re} \beta(X) < 0 \iff \operatorname{Re} \beta(X') < 0$$

for all $\beta \in \Delta(X)$, and the complex 1-dimensional subspace $\{tX'; t \in \mathbb{C}\} \subset \mathfrak{g}_{\mathbb{C}}$ is the Lie algebra of a closed algebraic subgroup $\mathbb{C}^{\times}(X') \subset G_{\mathbb{C}}$ isomorphic to \mathbb{C}^{\times} . We fix an embedding of $\mathbb{C}^{\times}(X') \simeq \mathbb{C}^{\times}$ into \mathbb{C} so that the tangent map sends $X' \in T_e(\mathbb{C}^{\times}(X'))$ into an element with nonnegative real part. This embedding allows us to take limits as $z \in \mathbb{C}^{\times}(X')$ approaches to zero, and for each $x \in M$ with $X_M(x) = 0$ we set

$$\tilde{O}_x = \{y \in M; \lim_{z \rightarrow 0} z^{-1} \cdot y = x\}.$$

The sets \tilde{O}_x are smooth locally closed algebraic subvarieties of M .

Then

$$\begin{aligned} \{x \in \mathcal{B} \times M; X_{\mathcal{B} \times M}(x) = 0\} \\ &= \{x \in \mathcal{B}; X_{\mathcal{B}}(x) = 0\} \times \{x \in M; X_M(x) = 0\} \\ &= \{\mathfrak{b}_1, \dots, \mathfrak{b}_{|W|}\} \times \{x \in M; X_M(x) = 0\} \end{aligned}$$

is the set of zeroes of the vector field $X_{\mathcal{B} \times M}$, and

$$\mathcal{B} \times M = \coprod_{\substack{k=1, \dots, |W| \\ \{x \in M; X_M(x)=0\}}} O_k \times \tilde{O}_x$$

is the Bialynicki-Birula decomposition of $\mathcal{B} \times M$ into attracting sets (relative to X). Applying the integral localization formula to the integral (18) yields

$$\frac{i^n}{(2\pi i)^{d+n}} \int_{\Lambda} \left(\int_{\mathfrak{g}_{\mathbb{R}}} \tilde{p}^* e^{\langle X, \mu_{\mathcal{B}}(\zeta) - \lambda_x \rangle + \sigma_{\mathcal{B}} + \tau_{\lambda}} \wedge \tilde{q}^* \text{Thom}_{T^*M} \wedge \varphi(X) \right)_{[2(d+n)]} = \int_{\mathfrak{g}_{\mathbb{R}}} \tilde{F}_{\theta} \varphi$$

for any $\varphi \in \Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$ with $\text{supp } \varphi \subset \mathfrak{g}'_{\mathbb{R}}$, where the function \tilde{F}_{θ} is

$$(26) \quad \tilde{F}_{\theta}(X) = (-1)^d \sum_{k=1}^{|W|} \sum_{\{x \in M; X_M(x)=0\}} \tilde{m}_{k,x}(X) \frac{e^{-\langle X, \lambda_{\mathfrak{b}_k} \rangle}}{\prod_{\alpha \in \Psi_{\mathfrak{b}_k}^+(X)} \alpha(X)},$$

and

$$\begin{aligned} \tilde{m}_{k,x}(X) &= \chi(\mathcal{B} \times M, (\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))_{O_k \times \tilde{O}_x}) \\ &= \chi(\mathcal{B} \times M, (j_{O_k \times \tilde{O}_x \hookrightarrow \mathcal{B} \times M})! \circ (j_{O_k \times \tilde{O}_x \hookrightarrow \mathcal{B} \times M})^*(\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))). \end{aligned}$$

We want to show that $\tilde{F}_{\theta}(X) = F_{\theta}(X)$. We have:

$$\begin{aligned} \sum_{\{x \in M; X_M(x)=0\}} \tilde{m}_{k,x}(X) &= \sum_{\{x \in M; X_M(x)=0\}} \chi(\mathcal{B} \times M, (\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))_{O_k \times \tilde{O}_x}) \\ &= \chi(\mathcal{B} \times M, (\mathbb{D}_{\mathcal{B} \times M}(\tilde{\mathcal{S}}))_{O_k \times M}) = m_{\mathfrak{b}_k}(X). \end{aligned}$$

Combining this with the equations (20) and (26) we obtain

$$\tilde{F}_{\theta}(X) = (-1)^d \sum_{k=1}^{|W|} m_{x_k} \frac{e^{-\langle X, \lambda_{\mathfrak{b}_k} \rangle}}{\prod_{\alpha \in \Psi_{\mathfrak{b}_k}^+(X)} \alpha(X)} = F_{\theta}(X),$$

which proves (18) at least when $\text{supp}(\varphi) \subset \mathfrak{g}'_{\mathbb{R}}$. Since the function F_{θ} is known to be a character of some representation (namely (11)), it is locally L^1 and the main result of [L3] now proves (18) for all $\varphi \in \Omega_c^{top}(\mathfrak{g}_{\mathbb{R}})$.

6. AN EXAMPLE ON THE ENHANCED FLAG VARIETY

The purpose of this section is to show how Theorem 1 can be applied to smooth quasi-projective varieties which are not necessarily projective. We let our group $G_{\mathbb{R}}$ be $GL(N, \mathbb{R}) \subset GL(N, \mathbb{C}) = G_{\mathbb{C}}$, for some $N \in \mathbb{N}$; we denote by $B_{\mathbb{C}}$ the group of all invertible lower-triangular matrices and by $N_{\mathbb{C}} \subset B_{\mathbb{C}}$ the group of lower-triangular unipotent matrices, we set $B_{\mathbb{R}} = B_{\mathbb{C}} \cap G_{\mathbb{R}}$ and $N_{\mathbb{R}} = N_{\mathbb{C}} \cap G_{\mathbb{R}}$. We will be interested in the complex homogeneous space $Z_{\mathbb{C}} = G_{\mathbb{C}}/N_{\mathbb{C}}$ and its real submanifold $Z_{\mathbb{R}} = G_{\mathbb{R}}/N_{\mathbb{R}} \subset Z_{\mathbb{C}}$. The space $Z_{\mathbb{C}}$ can be regarded as a fiber bundle over the flag variety $\mathcal{B} = G_{\mathbb{C}}/B_{\mathbb{C}}$ with fibers isomorphic to $H_{\mathbb{C}} =_{def} B_{\mathbb{C}}/N_{\mathbb{C}} \simeq (\mathbb{C}^{\times})^N$. It is easy to see that these fibers can be identified with each other in a canonical way. This allows us to think of $Z_{\mathbb{C}}$ as a trivial principal $H_{\mathbb{C}}$ -bundle which is also a $G_{\mathbb{C}}$ -equivariant fiber bundle, and the actions of $H_{\mathbb{C}}$ and $G_{\mathbb{C}}$ commute. W. Schmid and K. Vilonen call $Z_{\mathbb{C}}$ the *enhanced flag variety* of $\mathfrak{g}_{\mathbb{C}}$ [SchV2].

Let $\mathcal{S}' = (j_{Z_{\mathbb{R}} \hookrightarrow Z_{\mathbb{C}}})_*(\mathbb{C}_{Z_{\mathbb{R}}})$ denote the sheaf on $Z_{\mathbb{C}}$ which is the direct image of the constant sheaf on $Z_{\mathbb{R}}$. Fix a $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ and denote by $\ker \chi_{\lambda-\rho} \subset \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ the kernel of the character $\chi_{\lambda-\rho} : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathbb{C}$, and take a $\mathcal{D}_{Z_{\mathbb{C}}}$ -module

$$\mathfrak{M}' = \mathcal{D}_{Z_{\mathbb{C}}} / \gamma_{Z_{\mathbb{C}}}(\ker \chi_{\lambda-\rho}) \mathcal{D}_{Z_{\mathbb{C}}}.$$

Notice that by construction $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ acts on \mathfrak{M}' by character $\chi_{\lambda-\rho}$. We form a virtual representation of $G_{\mathbb{R}}$

$$(27) \quad \mathrm{RHom}_{\mathcal{D}_{Z_{\mathbb{C}}}}^{\mathrm{top}}(\mathfrak{M}' \otimes \mathcal{S}', \mathcal{O}_{Z_{\mathbb{C}}^{\mathrm{an}}}).$$

The composition factors of this representation are known to be the principal series representations of $G_{\mathbb{R}}$ which have the same character on $\mathfrak{g}_{\mathbb{R}}$. We will obtain an integral formula for the character of this representation. Of course, the homogeneous space $Z_{\mathbb{C}}$ is not compact, and in order to apply Theorem 1 we need to compactify $Z_{\mathbb{C}}$ first.

Recall that $Z_{\mathbb{C}}$ is a principal $(H_{\mathbb{C}}, G_{\mathbb{C}})$ -equivariant fiber bundle with commuting $H_{\mathbb{C}}$ - and $G_{\mathbb{C}}$ -actions. Hence we can embed $H_{\mathbb{C}} =_{\mathrm{def}} B_{\mathbb{C}}/N_{\mathbb{C}} \simeq (\mathbb{C}^{\times})^N$ into the vector space \mathbb{C}^N and form a $G_{\mathbb{C}}$ -equivariant vector bundle $\tilde{Z}_{\mathbb{C}} \simeq \mathbb{C}^N \times \mathcal{B}$ with an embedding $Z_{\mathbb{C}} \hookrightarrow \tilde{Z}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ -equivariant fiber bundles over \mathcal{B} . Next we form a line bundle $L_0 = \mathbb{C} \times \mathcal{B}$ over \mathcal{B} with trivial $G_{\mathbb{C}}$ -action on the first factor and form a projectivization $M =_{\mathrm{def}} \mathbb{P}(\tilde{Z}_{\mathbb{C}} \oplus L_0)$. Then M is a smooth complex projective variety, $G_{\mathbb{C}}$ acts on M algebraically, and M contains $Z_{\mathbb{C}}$ as a dense open subset. Define

$$\mathfrak{M} = \mathcal{D}_M / \gamma_M(\ker \chi_{\lambda-\rho}) \mathcal{D}_M \quad \text{and} \quad \mathcal{S} = (j_{Z_{\mathbb{C}} \hookrightarrow M})! \mathcal{S}' = (j_{Z_{\mathbb{R}} \hookrightarrow M})!(\mathbb{C}_{Z_{\mathbb{R}}}).$$

For convenience we restate Theorem 5.12 of [KaSchm]:

Theorem 2. *Let $f : X \rightarrow Y$ be a $G_{\mathbb{C}}$ -equivariant morphism between complex algebraic, quasi-projective $G_{\mathbb{C}}$ -manifolds X, Y . If f is projective, there exists an isomorphism*

$$\mathrm{RHom}_{\mathcal{D}_X}^{\mathrm{top}}(\mathfrak{M} \otimes f_{\mathrm{an}}^{-1} \mathcal{T}, \mathcal{O}_{X^{\mathrm{an}}}) \simeq \mathrm{RHom}_{\mathcal{D}_Y}^{\mathrm{top}}\left(\int_f \mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{Y^{\mathrm{an}}}\right)[- \dim X/Y],$$

functorially in $\mathcal{T} \in D_{G_{\mathbb{R}}, \mathbb{R}-c}^b(\mathbb{C}_Y)$ and $\mathfrak{M} \in D_{G_{\mathbb{C}}, \mathrm{coh}}^b(\mathcal{D}_X)$ (the bounded derived category of $\mathrm{Mod}_{\mathrm{quasi}-G_{\mathbb{C}}-eq}^{\mathrm{coh}}(\mathcal{D}_X)$). If f is smooth,

$$\mathrm{RHom}_{\mathcal{D}_X}^{\mathrm{top}}(Lf^* \mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{X^{\mathrm{an}}}) \simeq \mathrm{RHom}_{\mathcal{D}_Y}^{\mathrm{top}}(\mathfrak{M} \otimes R(f_{\mathrm{an}})! \mathcal{T}, \mathcal{O}_{Y^{\mathrm{an}}})[-2 \dim X/Y],$$

functorially in $\mathfrak{M} \in D_{G_{\mathbb{C}}, \mathrm{coh}}^b(\mathcal{D}_Y)$ and $\mathcal{T} \in D_{G_{\mathbb{R}}, \mathbb{R}-c}^b(\mathbb{C}_X)$.

Applying the second part of this theorem to the open inclusion $Z_{\mathbb{C}} \hookrightarrow M$ we can rewrite our virtual representation (27) as

$$\mathrm{RHom}_{\mathcal{D}_{Z_{\mathbb{C}}}}^{\mathrm{top}}(\mathfrak{M}' \otimes \mathcal{S}', \mathcal{O}_{Z_{\mathbb{C}}^{\mathrm{an}}}) \simeq \mathrm{RHom}_{\mathcal{D}_M}^{\mathrm{top}}(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{M^{\mathrm{an}}}).$$

The \mathcal{D}_M -module \mathfrak{M} is $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and lies in $\mathrm{Mod}_{\mathrm{quasi}-G_{\mathbb{C}}-eq}^{\mathrm{coh}, \lambda-\rho}(\mathcal{D}_M)$ essentially by construction. It follows from the Bruhat decomposition of $G_{\mathbb{C}}$ that $B_{\mathbb{C}}$ acts on M with finitely many orbits; this implies that \mathfrak{M} is admissible. It is easy to see that each maximal torus $T_{\mathbb{C}}$ acts on M with finitely many fixed points. Therefore Theorem 1 applies here and we obtain an integral formula for the character of the virtual representation (27). This is probably the most complicated formula for the principal series character there is.

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